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Analytical determination of the temperature distribution and Nusselt numbers in rectangular ducts with constant axial heat flux

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Abstract

A rigorous solution is obtained for the temperature field and the Nusselt numbers in the fully developed thermal region of rectangular ducts, wherein a laminar fully developed velocity profile occurs. The H1 thermal boundary condition, constant axial wall heat flux with a constant peripheral wall temperature, has been examined. The two dimensional (2D) temperature distribution and the Nusselt numbers are calculated as functions of the aspect ratio. The results, in terms of 2D-temperature profiles and Nusselt numbers, are presented and discussed in tables and graphs, considering all the possible combinations of heated and adiabatic walls of the rectangular cross section. A comparison with the numerically evaluated H1 Nusselt numbers found in literature is presented. The present analytical results are a powerful tool to test the commercial or self-produced thermal fluid-dynamic codes which permit the investigation of the internal forced convection of incompressible flows. © 1999 Elsevier Science Ltd. All rights reserved.

Keywords: Rectangular duct; Laminar forced convection; Analytical results

1. Introduction

The analysis of the thermal behaviour of Newtonian laminar flows through rectangular ducts is a topic of special interest in compact and micro-heat exchangers; in fact in these components the hydraulic diameter of the channels is so small that the flow regime is usually laminar. Rectangular ducts are employed in the tube-fin heat exchangers and in the plate-fin heat exchangers where this geometry provides a large surface-area-to-volume ratio and enhances the heat transfer coefficients. The micro-heat exchangers are an efficient means for removing high heat fluxes with relatively small temperature gradients. Most compact and micro-heat exchanger studies have focused on a conduit with a rectangular cross-section in the flow direction [1,2]. The theoretical analysis of the thermal behaviour of rectangular ducts is more complex and rarer than in the case of circular pipe flow. In fact, the investigation of rectangular ducts is very complicated because it requires a two-dimensional (2D) analysis. Generally, the thermal boundary conditions are also complex because there are many ways of imposing different temperatures or heat fluxes on the four wetted sides. For example, in a compact or micro-heat exchanger it can occur only one or two sides of the section are heated with a uniform heat flux whereas the other sides are insulated [2]. A clear understanding of the thermal boundary conditions is essential. Recent

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Nomenclature

a, b	longer and shorter sides, respectively, of the rectangular cross section (m)
A	number defined in Eq. (7)
c_p	fluid specific heat (J kg^{-1} K^{-1})
D_h	hydraulic diameter of the duct $2ab/(a+b)$ (m)
g_i	coefficients defined in Eq. (21)
h	heat transfer coefficient (W m^{-2} K^{-1})
j, k, l, n, m, p	summation integer indices
K	fluid thermal conductivity (W m^{-1} K^{-1})
L_h	heated perimeter length (m)
L_h^+	dimensionless heated perimeter length L_h/a
Nu	Nusselt number hD_h/K
q'	thermal power per unit of length (W m^{-1})
$t_{n,m}$	coefficients defined in Eq. (13)
$T^*(\cdot)$	dimensionless fluid temperature defined in Eq. (4)
$T(\cdot)$	dimensionless fluid temperature defined in Eq. (8)
$v_{n,m}$	coefficients defined in Eq. (7)
$u(\cdot)$	fluid velocity (m s^{-1})
$V(\cdot)$	dimensionless fluid velocity
W	average fluid velocity (m s^{-1})
x, y, z	dimensionless rectangular Cartesian co-ordinates
α	fluid thermal diffusivity (m^{2}s^{-1})
β	aspect ratio $b/a < 1$
ρ	fluid density (kg m^{-3})
$\theta(\cdot)$	fluid temperature (K)
ξ, η, ζ	Cartesian co-ordinates (m)

Subscripts

b	bulk quantity
w	quantity evaluated at the wall
0	inlet quantity

reviews [3,4] have proposed a systematic exposition of the main classes of boundary conditions. For non-circular ducts heated, for example, with an electric resistance, all with negligible normal wall thermal resistance, it is possible to consider two cases:

(a) for highly conductive materials (e.g., copper, aluminium) the axial wall heat flux may be considered to be constant with uniform peripheral wall temperature (H1 boundary condition);
(b) for very low conductive materials (e.g., glass-ceramic, teflon) with the duct having uniform wall thickness, the axial wall heat flux can be fixed as constant with a uniform peripheral wall heat flux (H2 boundary condition).

Moreover, many different situations can be considered, assuming a particular condition for every side of the rectangle; in the literature, eight classic thermal versions are proposed for the H1 and H2 problem. For any boundary condition, extensive numerical, analytical and experimental studies have been carried out for laminar fully developed flow.

For the H2 boundary condition, an analytical solution for any thermal version considered was published by Spiga and Morini [5]. The temperatures and the Nusselt numbers are analytically predicted in Ref. [5] as a function of the aspect ratio of the rectangular duct under the assumption of a steady-state laminar flow, fully developed both hydrodynamically and thermally.

The Nusselt numbers in laminar, fully developed flow under H1 boundary conditions have been calculated numerically by many authors. Glaser [6] uses a finite difference technique to study the temperature distribution in a square channel subjected to H1 boundary conditions. Clark and Kays [7] complete that work by determining the Nusselt numbers in rectangular channels with aspect ratios equal to 0 (slab), 0.25, 1/3, 0.5 and 1/1.4. They use a discretized grid of $10 \times 10\beta$ meshes, where β is the aspect ratio of the rectangular duct investigated. Miles and Shin [8] increase the grid meshes to $40 \times 40\beta$. Schmidt and Newell [9] calculate the Nusselt number by a finite difference method in the eight possible combinations of adiabatic and heated sides. Their results, reported as a function of the aspect ratio β, are corrected by Shah and London [4], who calculate the Nusselt number by using the hydraulic diameter instead of the heated perimeter. Chandrupatla and Sastri [10] obtain the fully developed H1 Nusselt number for a square duct as a particular case of the numerical solutions for laminar heat transfer of a non-Newtonian fluid. Montgomery and Wilbulswas [11] and Lyczkowski et al. [12] solved the thermal entry length problem for a square duct subjected to the H1 boundary condition by using the explicit finite difference method. The only analytical solution for evaluating the temperature profile in a rectangular duct for the H1 boundary condition is due to Marco and Han [13] who analyse the H1 problem with all sides heated by applying the formal analogy that exists between this thermal problem and the shear stress problem in a thin slab subjected to uniform load on the perimeter.

The aim of this paper is the rigorous determination of the temperature profile and the Nusselt numbers of a fluid with laminar fully developed velocity profile through rectangular ducts for the H1 boundary conditions for any combination of heated and adiabatic sides.

2. Basic equations

2.1. Energy equation

Consider a steady laminar flow in the thermally developed region of a rectangular duct with axially unchanging cross-section. A Cartesian system of coordinates ξ, η, ζ is assumed, with its origin in the left bottom corner of the inlet rectangular cross section (η along the short side b, ζ perpendicular to the cross section). The fluid has a laminar fully developed profile of velocity $u(\xi,\eta)$ and a uniform inlet temperature $\theta_0(\zeta=0)$. Under the assumption of constant fluid properties and neglecting axial thermal conduction, natural convection, viscous dissipation and internal energy sources, with rigid and non-porous duct walls, the differential steady state energy equation may be written as:

$$\frac{\partial^2\theta}{\partial\xi^2}+\frac{\partial^2\theta}{\partial\eta^2}=\frac{u(\xi,\eta)}{\alpha}\frac{\partial\theta}{\partial\zeta} \tag{1}$$

In order to solve Eq. (1) an energy balance between section ζ and $\zeta+\mathrm{d}\zeta$ gives the axial variation of the fluid bulk temperature θ_b for the H1 boundary condition:

$$\frac{\partial\theta_\mathrm{b}}{\partial\zeta}=\frac{q'}{\rho c_p Wab} \tag{2}$$

where q' is the thermal power per unit of length imposed on the heated walls of the rectangular duct and W is the average velocity. In the fully developed thermal region of a heated duct the temperature profile continues to change with ζ but the 'relative temperature shape' of the profile no longer changes. It is possible to demonstrate that, in the thermal fully developed region for the H1 boundary condition, the following ensures:

$$\frac{\partial\theta_\mathrm{b}}{\partial\zeta}=\frac{\partial\theta}{\partial\zeta} \tag{3}$$

It is suitable to introduce the dimensionless quantities:

$$x=\frac{\xi}{a},\quad 0\le x\le 1;\quad y=\frac{\eta}{a},\quad 0\le y\le\beta=\frac{b}{a};$$
$$T^*=\frac{K(\theta-\theta_0)}{q'},\quad V(x,y)=\frac{u}{W} \tag{4}$$

Consequently, the dimensionless energy balance equation is readily obtained in the following forms for the H1 problem examined:

$$\frac{\partial^2T^*}{\partial x^2}+\frac{\partial^2T^*}{\partial y^2}=\frac{V(x,y)}{\beta} \tag{5}$$

where the dimensionless velocity distribution in laminar flow through rectangular ducts is well known (see Spiga and Morini [14]):

$$V(x,y)=\sum_{n=1,\mathrm{odd}}^{\infty}\sum_{m=1,\mathrm{odd}}^{\infty}v_{n,m}\,\sin(n\pi x)\sin\left(\frac{m\pi y}{\beta}\right) \tag{6}$$

The series coefficients $v_{n,m}$ are:

$$v_{n,m}=\frac{\pi^2}{4mn\left(\beta^2n^2+m^2\right)}\frac{1}{\sum_{i=1,\mathrm{odd}}^{\infty}\sum_{j=1,\mathrm{odd}}^{\infty}\frac{1}{i^2j^2\left(\beta^2i^2+j^2\right)}}$$
$$=\frac{1}{A(\beta)mn\left(\beta^2n^2+m^2\right)} \tag{7}$$

where the number $A(\beta)$ depends only on the aspect ratio of the duct [5] and can be approximated by the third-order polynomial: $0.5059-0.3022\beta-0.00642\beta^2+0.0747\beta^3$.

2.2. Thermal boundary conditions

In this paper, the eight thermal versions proposed in the literature for the H1 problem will be considered; the following nomenclature is usually assumed in the analysis of rectangular ducts for those eight versions: 4: four constant wall temperatures; 3L: three constant wall temperatures and one adiabatic short side; 3S: three constant wall temperatures and one adiabatic long side; 2L: two constant wall temperatures and two adiabatic short sides; 2S: two constant wall temperatures and two adiabatic long sides; 2C: one short and one long constant wall temperatures (corner version); 1L: one constant wall temperature long side; 1S: one constant wall temperature short side.

The H1 boundary condition states that the wall temperature $T_\mathrm{w}(z)$ is uniform on the heated length of a rectangular perimeter and that it increases linearly with the z longitudinal coordinate (Marco and Han [13]). Therefore, the temperature field can be written as:

$$T^*(x,y,z)=T_\mathrm{w}(z)+T(x,y) \tag{8}$$

Table 1
Coefficients d_{Ni} for the H1 problem

Version	d_{1x}	d_{2x}	d_{3x}	d_{4x}	d_{1y}	d_{2y}	d_{3y}	d_{4y}
1L	0	1	0	1	1 (or 0)	0 (or 1)	0 (or 1)	1 (or 0)
1S	1 (or 0)	0 (or 1)	0 (or 1)	1(or 0)	0	1	0	1
2L	0	1	0	1	1	0	1	0
2S	1	0	1	0	0	1	0	1
2C	1 (or 0)	0 (or 1)	0 (or 1)	1 (or 0)	1 (or 0)	0 (or 1)	0 (or 1)	1 (or 0)
3L	1 (or 0)	0 (or 1)	0 (or 1)	1 (or 0)	1	0	1	0
3S	1	0	1	0	1 (or 0)	0 (or 1)	0 (or 1)	1 (or 0)
4	1	0	1	0	1	0	1	0

It is possible to use the energy equation (5) to determine the part of the temperature distribution that does not depend on z; it can be observed that on the heated perimeter $T(x,y)$ vanishes. In order to find the temperature field $T(x,y)$ for each H1 thermal version, the boundary conditions that one combines with the energy equation can be written as:

$$\begin{cases} d_{1x}T(0,y) + d_{2x}\left.\dfrac{\mathrm{d}T}{\mathrm{d}x}\right|_{x=0} = 0 \\ d_{3x}T(1,y) + d_{4x}\left.\dfrac{\mathrm{d}T}{\mathrm{d}x}\right|_{x=1} = 0 \\ d_{1y}T(x,0) + d_{2y}\left.\dfrac{\mathrm{d}T}{\mathrm{d}y}\right|_{y=0} = 0 \\ d_{3y}T(x,\beta) + d_{4y}\left.\dfrac{\mathrm{d}T}{\mathrm{d}y}\right|_{y=\beta} = 0 \end{cases} \quad (9)$$

The coefficients d_{Ni} depend on the specified combination of heated and adiabatic walls, imposed by the boundary conditions. With reference to the eight thermal versions considered, the values assumed by the constants d_{Ni} are shown in Table 1 for the H1 problem.

2.3. Bulk temperature and Nusselt number

The knowledge of both the temperature and velocity distribution over a cross section of the rectangular duct allows the determination of the bulk temperature:

$$T_\mathrm{b} = \frac{1}{\beta}\int_0^1\int_0^\beta T(x,y)V(x,y)\,\mathrm{d}x\,\mathrm{d}y \quad (10)$$

Finally, the Nusselt number can be obtained by an energy balance on the heated perimeter of the rectangular duct; its expression is:

$$Nu = -\frac{2\beta}{L_\mathrm{h}^{+}(1+\beta)T_\mathrm{b}} \quad (11)$$

where L_h^{+} is the dimensionless heated perimeter defined as L_h/a.

3. The solution procedure: the finite-integral transform method

The differential problem defined by Eq. (5), subject to the boundary conditions specified for all the thermal versions that have been considered (Eq. (9)), is linear and its solution can be tackled by the finite-integral transform technique.

In order to solve the temperature problem given by Eq. (5) with the eigenfunction expansion technique, the appropriate eigenvalue problem is taken as:

$$\begin{cases} \dfrac{\mathrm{d}\Phi_{i,n}}{\mathrm{d}i} + \lambda_{i,n}^2\Phi_{i,n} = 0 \\ d_{1i}\Phi_{i,n}(0) + d_{2i}\left.\dfrac{\mathrm{d}\Phi_{i,n}}{\mathrm{d}i}\right|_0 = 0 \\ d_{3i}\Phi_{i,n}(\delta) + d_{4i}\left.\dfrac{\mathrm{d}\Phi_{i,n}}{\mathrm{d}i}\right|_\delta = 0 \quad \text{with } \delta = \begin{cases} 1 & \text{if } i = x \\ \beta & \text{if } i = y \end{cases} \end{cases} \quad (12)$$

where i is equal to x or y and n is the order of the gen-

Table 2
Eigenvalues and eigenfunctions for the H1 problem

Version	λ_{xn}	$\lambda_{y,m}$	$\Phi_{x,n}$	$\Phi_{y,m}$
1L	$n\pi$	$(2m-1)\dfrac{\pi}{2\beta}$	$\cos(\lambda_{x,n}x)$	$\sin(\lambda_{y,m}y)$
1S	$(2n-1)\dfrac{\pi}{2}$	$\dfrac{m\pi}{\beta}$	$\sin(\lambda_{x,n}x)$	$\cos(\lambda_{y,m}y)$
2L	$n\pi$	$\dfrac{m\pi}{\beta}$	$\cos(\lambda_{x,n}x)$	$\sin(\lambda_{y,m}y)$
2S	$n\pi$	$\dfrac{m\pi}{\beta}$	$\sin(\lambda_{x,n}x)$	$\cos(\lambda_{y,m}y)$
2C	$(2n-1)\dfrac{\pi}{2}$	$(2m-1)\dfrac{\pi}{\beta}$	$\sin(\lambda_{x,n}x)$	$\sin(\lambda_{y,m}y)$
3L	$(2n-1)\dfrac{\pi}{2}$	$\dfrac{m\pi}{\beta}$	$\sin(\lambda_{x,n}x)$	$\sin(\lambda_{y,m}y)$
3S	$n\pi$	$(2m-1)\dfrac{\pi}{2\beta}$	$\sin(\lambda_{x,n}x)$	$\sin(\lambda_{y,m}y)$
4	$n\pi$	$\dfrac{m\pi}{\beta}$	$\sin(\lambda_{x,n}x)$	$\sin(\lambda_{y,m}y)$

Table 3
The coefficients $t_{n,m}$ of Eq. (13): versions 1L, 1S, 2L, 2S

Version	$t_{n,m}$
1L	$-\frac{128\beta}{\pi^4 A(\beta)} \sum_{k=1,\mathrm{odd}}^{\infty} \sum_{j=1,\mathrm{odd}}^{\infty} \frac{\delta_n(-1)^{(m-1)/2}}{(4n^2\beta^2)} + m^2(\beta^2 k^2) + j^2(k^2) - n^2(4j^2) - m^2$ for $n = 0$ or even, and m odd
1S	$-\frac{128\beta}{\pi^4 A(\beta)} \sum_{k=1,\mathrm{odd}}^{\infty} \sum_{j=1,\mathrm{odd}}^{\infty} \frac{\delta_m}{} - 1^{(n-1)/2}(n^2\beta^2) + 4m^2(\beta^2 k^2) + j^2(4k^2) - n^2(j^2) - m^2$ for n odd, and $m = 0$ or even
2L	$-\frac{4\beta}{\pi^3 A(\beta)} \sum_{k=1,\mathrm{odd}}^{\infty} \frac{\delta_n}{m(n^2\beta^2)} + m^2(\beta^2 k^2) + m^2(k^2) - n^2$ for $n = 0$ or even and m odd
2S	$-\frac{4\beta}{\pi^3 A(\beta)} \sum_{j=1,\mathrm{odd}}^{\infty} \frac{\delta_m}{n(n^2\beta^2)} + m^2(\beta^2 n^2) + j^2(j^2) - m^2$ for n odd and $m = 0$ or even
	$\delta_i = 1$ if $i \neq 0$ $\delta_i = 1/2$ if $i = 0$

Table 4
The coefficients $t_{n,m}$ of Eq. (13): versions 2C, 3L, 3S, 4

2C	$-\frac{256\beta}{\pi^4 A(\beta)} \sum_{k=1,\mathrm{odd}}^{\infty} \sum_{j=1,\mathrm{odd}}^{\infty} \frac{(-1)^{n-1/2}(-1)^{m-1/2}}{(n^2\beta^2)} + m^2(\beta^2 k^2) + j^2(4k^2) - n^2(4j^2) - m^2$ for n and m odd
3L	$-\frac{32\beta}{\pi^3 A(\beta)} \sum_{k=1,\mathrm{odd}}^{\infty} \frac{-1^{(n-1)/2}}{m(n^2\beta^2)} + 4m^2(\beta^2 k^2) + m^2(4k^2) - n^2$ for n and m odd
3S	$-\frac{32\beta}{\pi^3 A(\beta)} \sum_{j=1,\mathrm{odd}}^{\infty} \frac{-1^{(m-1)/2}}{n(4n^2\beta^2)} + m^2(\beta^2 n^2) + j^2(4j^2) - m^2$ for n and m odd
4	$-\frac{\beta}{\pi^2 A(\beta) mn(n^2\beta^2)} + m^2 2$ for n and m odd

Table 5
The bulk temperature for the H1 problem: versions 1L, 1S, 2L, 2S

Version	T_b
1L	$-\frac{1024\beta}{\pi^6 A^2(\beta)} \sum_{n=0,\mathrm{even}}^{\infty} \sum_{m=1,\mathrm{odd}}^{\infty} \sum_{l=1,\mathrm{odd}}^{\infty} \sum_{p=1,\mathrm{odd}}^{\infty} \sum_{k=1,\mathrm{odd}}^{\infty} \sum_{j=1,\mathrm{odd}}^{\infty} \frac{1}{(\beta^2 k^2)} + j^2(k^2) - n^2(4j^2) - m^2 \frac{\delta_n}{(l^2\beta^2)}$ $+ p^2(4\beta^2 n^2) + m^2(l^2) - n^2(4p^2) - m^2$
1S	$-\frac{1024\beta}{\pi^6 A^2(\beta)} \sum_{n=1,\mathrm{odd}}^{\infty} \sum_{m=0,\mathrm{even}}^{\infty} \sum_{l=1,\mathrm{odd}}^{\infty} \sum_{p=1,\mathrm{odd}}^{\infty} \sum_{k=1,\mathrm{odd}}^{\infty} \sum_{j=1,\mathrm{odd}}^{\infty} \frac{1}{(\beta^2 k^2)} + j^2(4k^2) - n^2(j^2) - m^2 \frac{\delta_m}{(l^2\beta^2)} + p^2(\beta^2 n^2)$ $+ 4m^2(4l^2) - n^2(p^2) - m^2$
2L	$-\frac{4\beta}{\pi^4 A^2(\beta)} \sum_{n=0,\mathrm{even}}^{\infty} \sum_{m=1,\mathrm{odd}}^{\infty} \sum_{l=1,\mathrm{odd}}^{\infty} \sum_{k=1,\mathrm{odd}}^{\infty} \frac{\delta_n}{(\beta^2 k^2)} + m^2(k^2) - n^2 m^2(l^2\beta^2) + m^2(\beta^2 n^2) + m^2(l^2) - n^2$
2S	$-\frac{4\beta}{\pi^4 A^2(\beta)} \sum_{n=1,\mathrm{odd}}^{\infty} \sum_{m=0,\mathrm{even}}^{\infty} \sum_{p=1,\mathrm{odd}}^{\infty} \sum_{j=1,\mathrm{odd}}^{\infty} \frac{\delta_m}{(\beta^2 n^2)} + j^2(j^2) - m^2 n^2(n^2\beta^2) + m^2(\beta^2 n^2) + p^2(p^2) - m^2$
	$\delta_i = 1$ if $i \not\equiv 0$ $\delta_i = 1/2$ if $i = 0$

eral eigenvalue (λ) and of the general eigenfunction (Φ) that fulfils the Sturm–Liouville problem defined by Eq. (12), where the coefficients d_{Ni} assume, for each H1 thermal version considered, the values quoted in Table 1.

In heat conduction problems the infinite series of eigenvalues $\lambda_{i,n}$ and the related eigenfunctions $\Phi_{i,n}$ that resolve problem (12) are used frequently and their expressions can be found in all the textbooks on heat conduction (Cotta [15] and Ozisik [16]).

For the sake of completeness, Table 2 shows the eigenvalues and the eigenfunctions generated by problem (12) for all the thermal versions considered of the H1 problem.

Using the appropriate eigenfunctions (Table 2), the unknown temperature field is sought by resorting to a double series:

$$T(x,y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} t_{n,m}\Phi_{x,n}\Phi_{y,m} \tag{13}$$

The temperature distribution is defined, with the exception of an additive constant, if one determines the constants $t_{n,m}$. In order to obtain the solution to Eq. (5), the first step consists of substituting the laminar velocity distribution (Eq. (6)) in the right-hand side of Eq. (5) and of multiplying every term of the energy equation by $\Phi_{x,n}\Phi_{y,m}$ and by integrating over x, between 0 and 1, and over y, between 0 and β.

$$-\left(\lambda_{x,n}^2 + \lambda_{y,m}^2\right)N_{n,x}N_{m,y}t_{n,m} = \sum_{p=1,\text{odd}}^{\infty}\sum_{j=1,\text{odd}}^{\infty} \frac{v_{p,j}}{\beta}\psi_{p,n}\psi_{j,m} \tag{14}$$

where:

$$N_{n,i} = \int_0^{\delta} \Phi_{n,i}^2\,\mathrm{d}i, \qquad \psi_{k,l} = \int_0^{\delta} \sin\left(\frac{k\pi i}{\delta}\right)\Phi_{l,i}\,\mathrm{d}i \tag{15}$$

The integrals appearing in this procedure can be easily and patiently solved by the classic methods inasmuch as the eigenfunctions quoted in Table 2 are very simple trigonometric functions. Using Eq. (15), after some algebra it is possible to obtain the unknown coefficients $N_{n,i}$ and $\psi_{k,l}$ for all the thermal versions of the H1 problem examined; the temperature field for each H1 thermal version can be obtained by solving the algebraic equation (14) with respect to $t_{n,m}$ and using Eq. (13).

$$T(x,y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{p=1,\text{odd}}^{\infty}\sum_{j=1,\text{odd}}^{\infty} - \frac{1}{\left(\lambda_{x,n}^2 + \lambda_{y,m}^2\right)N_{n,x}N_{m,y}}\frac{v_{p,j}}{\beta}\psi_{p,n}\psi_{j,m}\Phi_{n,x}\Phi_{m,y} \tag{16}$$

From Eq. (10) one can calculate the bulk temperature for all the versions of the H1 problem. In fact, if one replaces the laminar velocity distribution (Eq. (6)) and the temperature field (Eq. (13)) in Eq. (10) it is easy to demonstrate that the bulk temperature may be written as:

$$T_\mathrm{b}(\beta) = \frac{1}{\beta}\sum_{l=1,\text{odd}}^{\infty}\sum_{k=1,\text{odd}}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} t_{n,m}v_{l,k}\psi_{l,n}\psi_{k,m} \tag{17}$$

Introducing this expression in Eq. (11) one obtains the exact expression of the Nusselt number for each version of the H1 problem considered.

3.1. Version 4

The proposed procedure can be tested for version 4 of the H1 problem. In this case the temperature field (Eq. (16)) can be expressed, by using the appropriate eigenfunctions and eigenvalues shown in Table 2, as:

Table 6
The bulk temperature for the H1 problem: versions 2C, 3L, 3S, 4

Version	T_b
2C	$-\frac{4096\beta}{\pi^6 A^2(\beta)}\sum_{n=1,\text{odd}}^{\infty}\sum_{m=1,\text{odd}}^{\infty}\sum_{l=1,\text{odd}}^{\infty}\sum_{p=1,\text{odd}}^{\infty}\sum_{k=1,\text{odd}}^{\infty}\sum_{j=1,\text{odd}}^{\infty}\frac{1}{(\beta^2 k^2)} + j^2(4k^2) - n^2(4j^2) - m^2(l^2\beta^2) + p^2(\beta^2 n^2)$
	$+\, m^2(4l^2) - n^2(4p^2) - m^2$
3L	$-\frac{64\beta}{\pi^4 A^2(\beta)}\sum_{n=1,\text{odd}}^{\infty}\sum_{m=1,\text{odd}}^{\infty}\sum_{j=1,\text{odd}}^{\infty}\sum_{k=1,\text{odd}}^{\infty}\frac{1}{m^2(\beta^2 k^2)} + m^2(4k^2) - n^2(j^2\beta^2) + m^2(\beta^2 n^2) + 4m^2(4j^2) - n^2$
3S	$-\frac{64\beta}{\pi^4 A^2(\beta)}\sum_{n=1,\text{odd}}^{\infty}\sum_{m=1,\text{odd}}^{\infty}\sum_{j=1,\text{odd}}^{\infty}\sum_{l=1,\text{odd}}^{\infty}\frac{1}{n^2(\beta^2 n^2)} + j^2(4j^2) - m^2(4n^2\beta^2) + m^2(\beta^2 n^2) + l^2(4l^2) - m^2$
4	$-\frac{\beta}{4\pi^2 A^2(\beta)}\sum_{n=1,\text{odd}}^{\infty}\sum_{m=1,\text{odd}}^{\infty}\frac{1}{n^2 m^2(\beta^2 n^2)} + m^2 3$

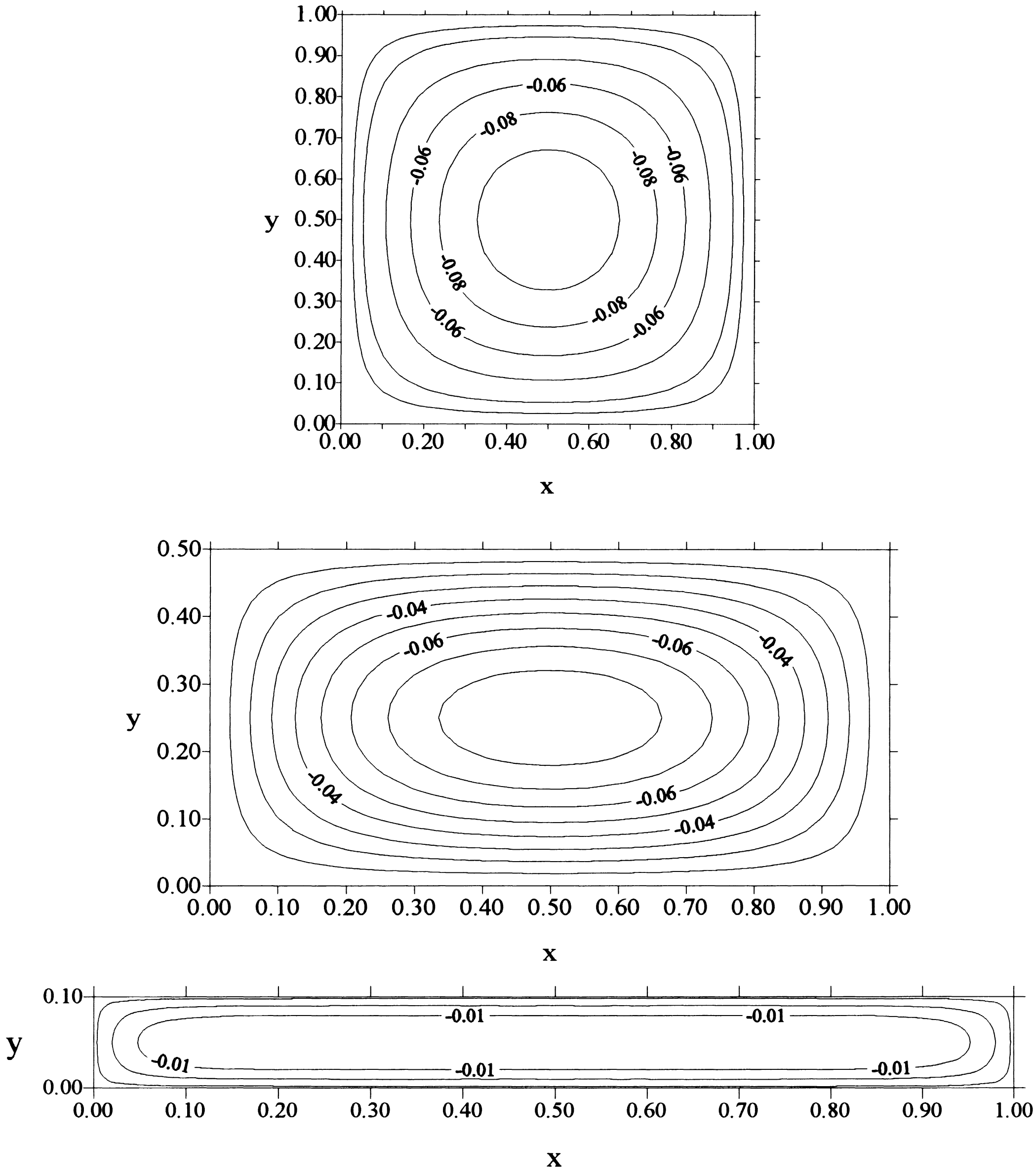

Fig. 1. Dimensionless temperature distributions in a rectangular duct with $\beta = 1$, 0.5 and 0.1, four sides heated (version 4).

$$T(x,y) = -\frac{\beta}{\pi^2} \sum_{n=1,\text{odd}}^{\infty} \sum_{m=1,\text{odd}}^{\infty} \frac{v_{n,m}}{\left(\beta^2 n^2 + m^2\right)} \times \sin(n\pi x) \sin\left(\frac{m\pi y}{\beta}\right) \tag{18}$$

In fact, in this case the eigenfunctions for the velocity and the temperature field are the same; for this reason the coefficients $\psi_{k,l}$ assume a simple value and some summations vanish. In fact it is easy to demonstrate that:

$$N_{n,i} = \int_0^\delta \Phi_{n,i}^2 \, \mathrm{d}i = \frac{\delta}{2},$$

$$\psi_{k,l} = \int_0^\delta \sin\left(\frac{k\pi i}{\delta}\right) \Phi_{l,i} \, \mathrm{d}i = \begin{cases} 0 & \text{if } k \neq l \\ \dfrac{\delta}{2} & \text{if } k = l \end{cases} \tag{19}$$

The bulk temperature in this case can be deducted by simplifying Eq. (17) by means of the results shown in Eq. (19):

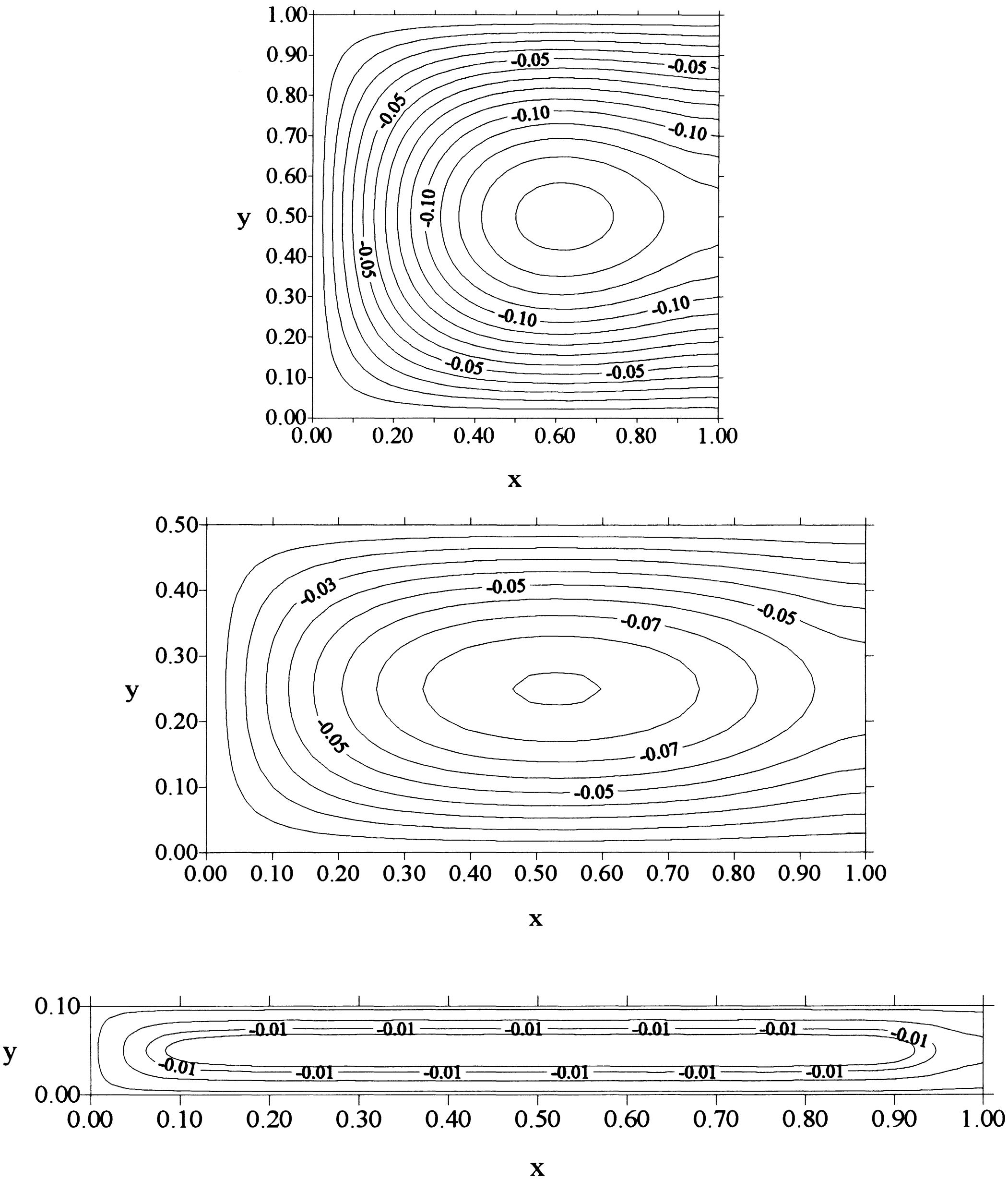

Fig. 2. Dimensionless temperature distributions in a rectangular duct with $\beta = 1$, 0.5 and 0.1, version 3L.

$$T_b(\beta) = -\frac{\beta}{4\pi^2} \sum_{n=1,\text{odd}}^{\infty} \sum_{m=1,\text{odd}}^{\infty} \frac{\nu_{n,m}^2}{\left(\beta^2 n^2 + m^2\right)} \tag{20}$$

If one combines this result with Eqs. (7) and (11) it is possible to demonstrate that the expression of the Nusselt number coincides with the expression quoted in the paper by Marco and Han [13] and used by Shah and London in [4]. In other words, the general procedure shown in this paper to determine the temperature field and the values assumed by the Nusselt number for each version of the H1 problem, makes it possible to obtain, as a particular case, the analytical results quoted in literature for the version 4.

3.2. The others versions

In Table 3 are quoted the expressions assumed by the $t_{n,m}$ coefficients for versions 1L, 1S, 2L and 2S of the H1 problem. For the versions 2C, 3L, 3S and 4 the $t_{n,m}$ coefficients are quoted in Table 4. Knowing the

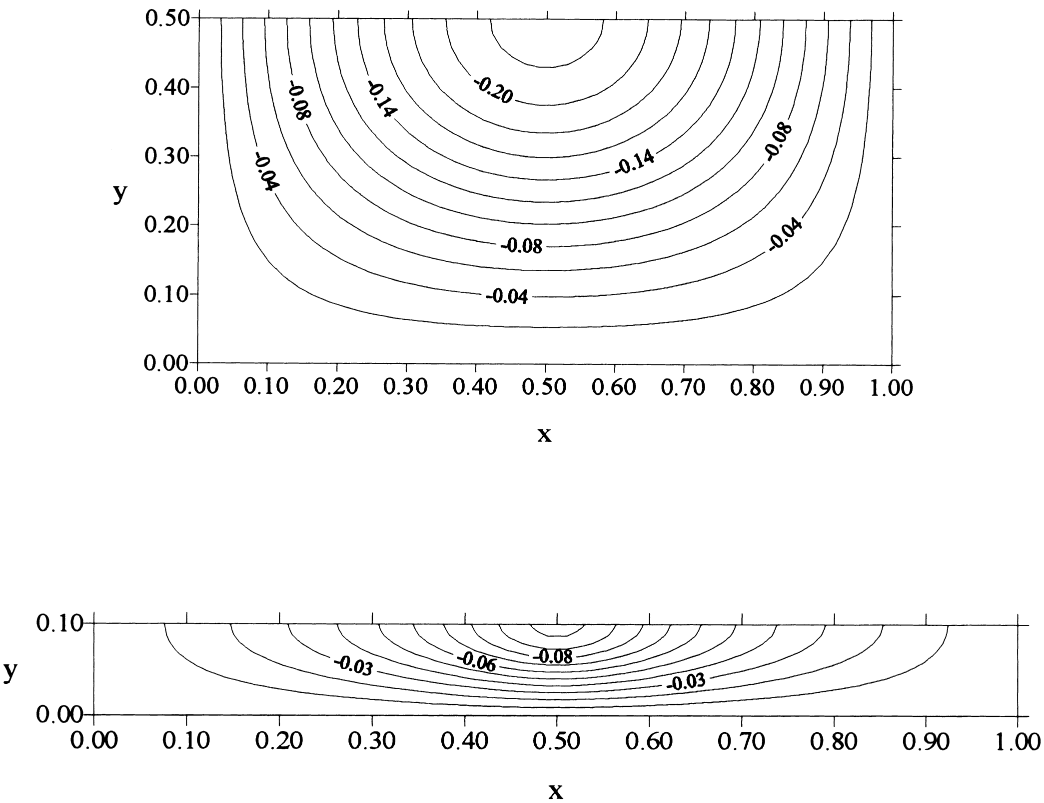

Fig. 3. Dimensionless temperature distributions in a rectangular duct with $\beta = 0.5$ and 0.1, version 3S.

$t_{n,m}$ coefficients, by means of Eq. (13) it is easy to calculate the temperature distribution inside the rectangular duct, to obtain the exact expression of the bulk temperature (defined in Eq. (10)) and hence, by using Eq. (11), to determine the Nusselt numbers for any combination of heated and adiabatic sides.

Table 7
Comparison of the analytical Nusselt numbers obtained in this paper with the numerical results found in literature: versions 2C, 3L, 3S, 4

β	*Nu*							
	4		3L		3S		2C	
	Present results and [4]	Ref. [9]	Present results	Refs. [4,9]	Present results	Refs. [4,9]	Present results	Refs. [4,9]
1	3.608	3.599	3.568	3.556	3.568	3.556	2.811	2.836
0.9	3.620	3.612	3.688	–	3.460	–	2.817	2.843
0.8	3.664	3.655	3.836	–	3.355	–	2.840	2.866
0.7	3.750	3.740	4.018	3.991	3.259	3.195	2.886	2.911
0.6	3.895	3.884	4.246	–	3.182	–	2.961	2.987
0.5	4.123	4.111	4.539	4.505	3.140	3.146	3.079	3.104
0.4	4.472	4.457	4.923	4.885	3.163	3.169	3.256	3.279
0.3	4.990	4.969	5.439	5.393	3.302	3.306	3.523	3.538
0.2	5.738	5.704	6.131	6.072	3.639	3.636	3.925	3.914
0.1	6.785	6.700	7.044	6.939	4.283	4.252	4.523	4.410
0	8.235	8.235	8.235	8.235	5.385	5.385	5.385	5.385

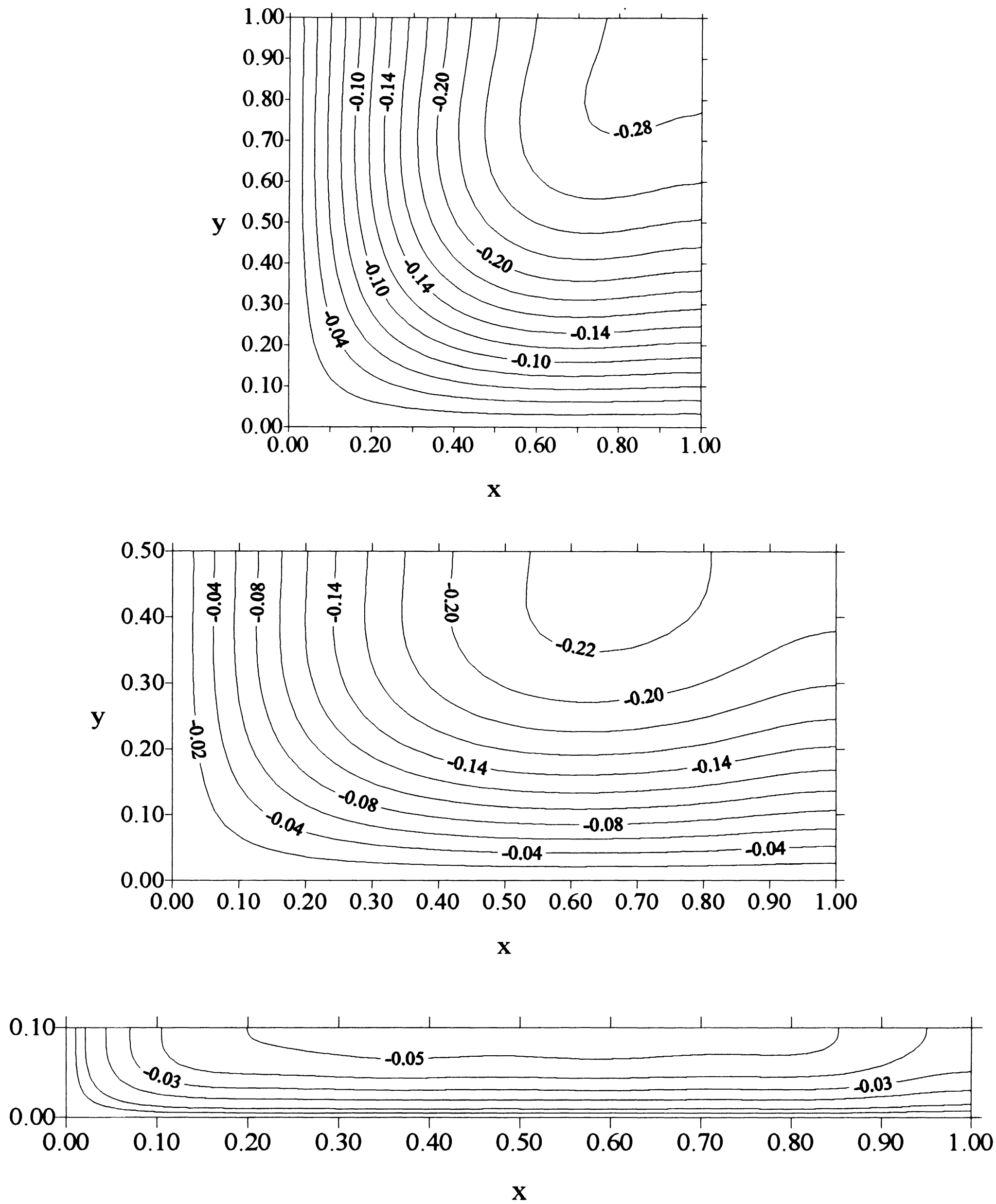

Fig. 4. Dimensionless temperature distributions in a rectangular duct with $\beta = 1$, 0.5 and 0.1, version 2C.

For the sake of completeness, the expressions of the bulk temperature for each version of the H1 problem are shown in Tables 5 and 6.

4. Results and discussion

The previous results have been worked out on a common PC using MATHEMATICA® 3.0: few program statements permit to do all the job. The availability of analytical expressions and the fast convergence of the multiple series invoked into the solution make the present technique quite effective and inexpensive in terms of computer time. The 2D temperature distribution through the rectangular duct is shown in Figs. 1–8, for different values of the aspect ratio (square duct, $\beta = 0.5$ and $\beta = 0.1$) and for all eight combinations of heated and adiabatic walls. In Fig. 1 the dimensionless fluid temperature field is sketched for the four heated walls (version 4), with

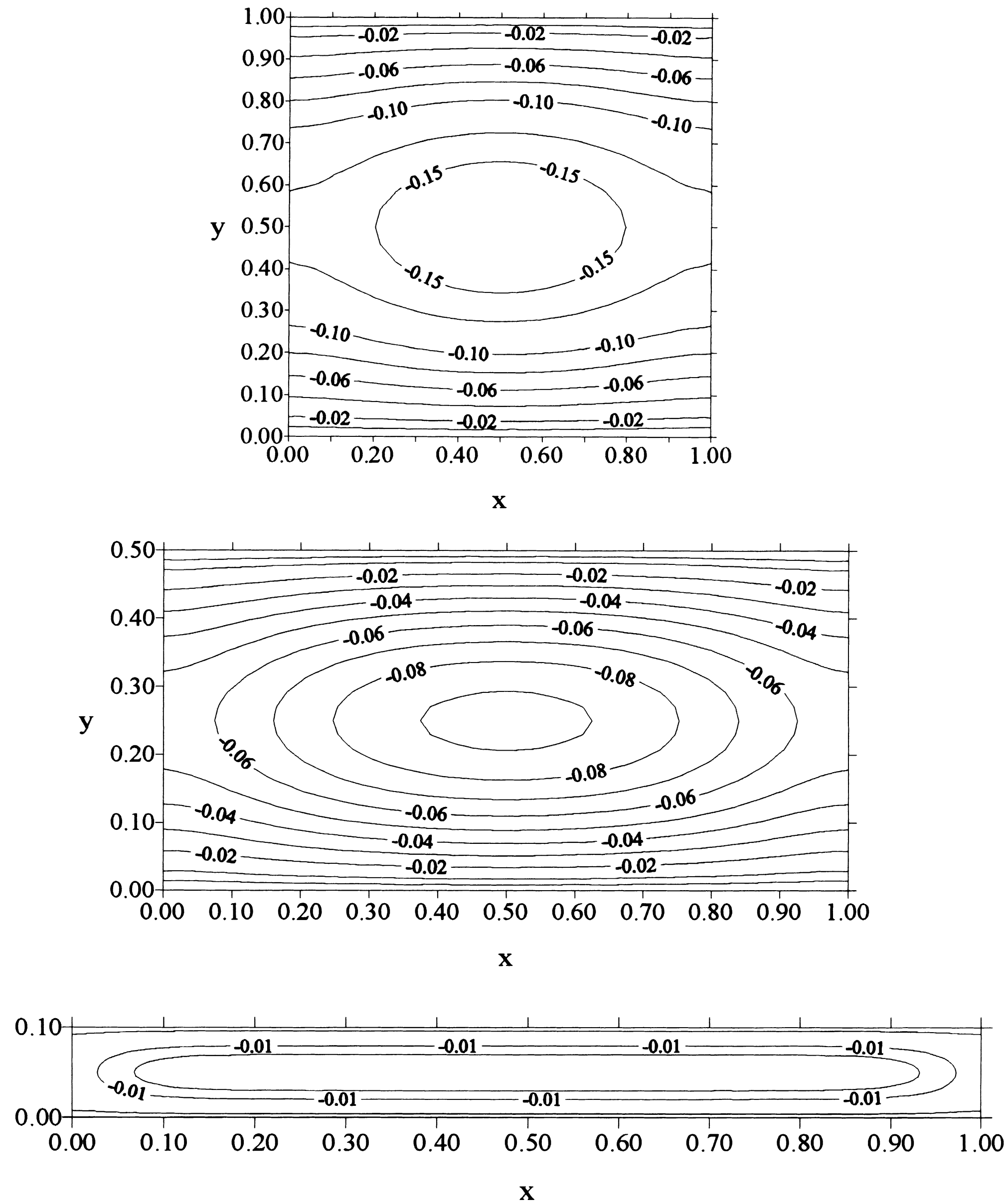

Fig. 5. Dimensionless temperature distributions in a rectangular duct with $\beta = 1$, 0.5 and 0.1, version 2L.

$\beta = 0.1$, 0.5 and 1. One can observe that the temperature assumes negative values in each internal cross section point and vanishes on the heated perimeter because in the present analysis only positive wall heat fluxes (i.e. from the wall to the fluid) have been considered. From Fig. 1 it turns out that the minimum fluid temperature occurs at the centre of the rectangular duct. The value of the point of minimum decreases with the aspect ratio until β reaches the value of 0.456; for lower aspect ratios the point of minimum increases [17]. In Fig. 2, for version 3L, the minimum fluid temperature lies on the straight line $y = \beta/2$ next to the adiabatic wall. The point of minimum departs from the adiabatic wall when the duct aspect ratio decreases. The solution to version 3S is shown in Fig. 3, for $\beta = 0.1$ and 0.5; obviously, for $\beta = 1$ (square duct) there is no difference between versions 3L and 3S; the same holds for versions 2L–2S and 1L–1S. It is interesting to observe that the minimum of the 3S temperature field, as opposed to version 3L, always occurs on the adiabatic wall. Fig. 4 refers to the 2C version; it is interesting to note that the minimum wall temperature

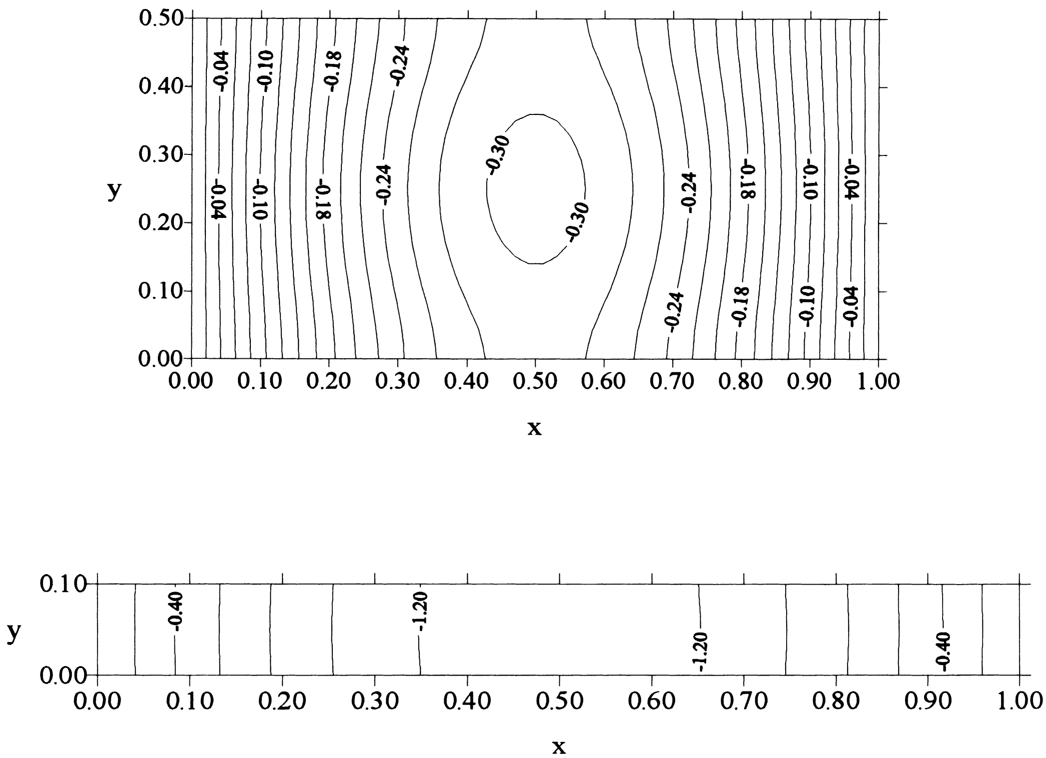

Fig. 6. Dimensionless temperature distributions in a rectangular duct with $\beta = 0.5$ and 0.1, version 2S.

is reached at the corner between the adiabatic sides only for the square duct. For $\beta < 1$ the temperature field is not symmetric with respect to the diagonal line between the adiabatic and the heated corners; it is evident that the minimum wall temperature lies on the adiabatic long side and departs from the adiabatic corner when the aspect ratio decreases. In Figs. 5 and 6 versions 2L and 2S are presented. The results show how, for small aspect ratios, the temperature distribution for version 2S tends to become 1D; in fact the

Table 8
Comparison of the analytical Nusselt numbers obtained in this paper with the numerical results found in literature: versions 1L, 1S, 2L, 2S

β	Nu							
	2L		2S		1L		1S	
	Present results	Refs. [4,9]	Present results	Refs. [4,9]	Present results	Refs. [4,9]	Present results	Refs. [4,9]
1	4.095	4.094	4.095	4.094	2.686	2.712	2.686	2.712
0.9	4.274	–	3.915	–	2.818	–	2.552	–
0.8	4.472	–	3.713	–	2.964	–	2.404	–
0.7	4.693	4.662	3.485	3.508	3.127	3.149	2.237	2.263
0.6	4.944	–	3.226	–	3.309	–	2.048	–
0.5	5.237	5.203	2.929	2.947	3.514	3.539	1.832	1.854
0.4	5.592	5.555	2.583	2.598	3.750	3.777	1.585	1.604
0.3	6.039	5.997	2.171	2.182	4.031	4.060	1.295	1.312
0.2	6.609	6.561	1.659	1.664	4.379	4.411	0.951	0.964
0.1	7.331	7.248	0.979	0.975	4.822	4.851	0.530	0.538
0	8.235	8.235	0	0	5.385	5.385	0	0

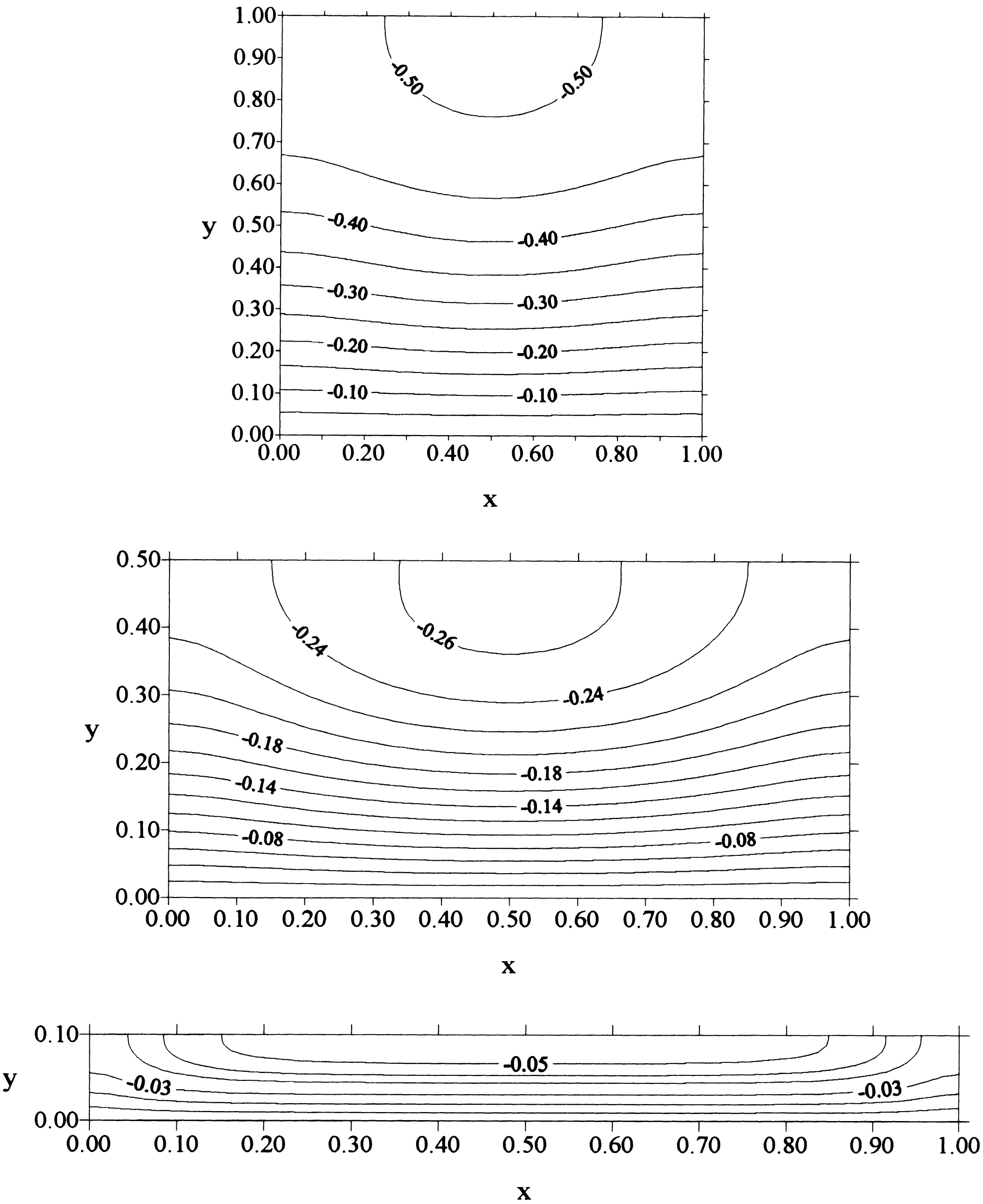

Fig. 7. Dimensionless temperature distributions in a rectangular duct with $\beta = 1$, 0.5 and 0.1, version 1L.

temperature variations along the y-direction are very small, as can be observed in Fig. 6 for $\beta = 0.1$. On the contrary, the border effect of the adiabatic sides is evident for version 2L with small aspect ratio too. The 1L and 1S temperature distributions are shown in Figs. 7 and 8; it is evident how the minimum wall temperature is located in the middle point of the long adiabatic side (if version 1L is considered) or of the short adiabatic side (if version 1S is considered). As for version 2S, when the aspect ratio of the duct decreases, the 1S temperature field through the rectangular channel tends to become 1D.

In Tables 7 and 8 are quoted the Nusselt numbers calculated by means of the procedure suggested in this paper as a function of the aspect ratio for the eight thermal versions considered. In these Tables the present analytical results are compared with the values quoted in the paper by Shah and London [4] and due

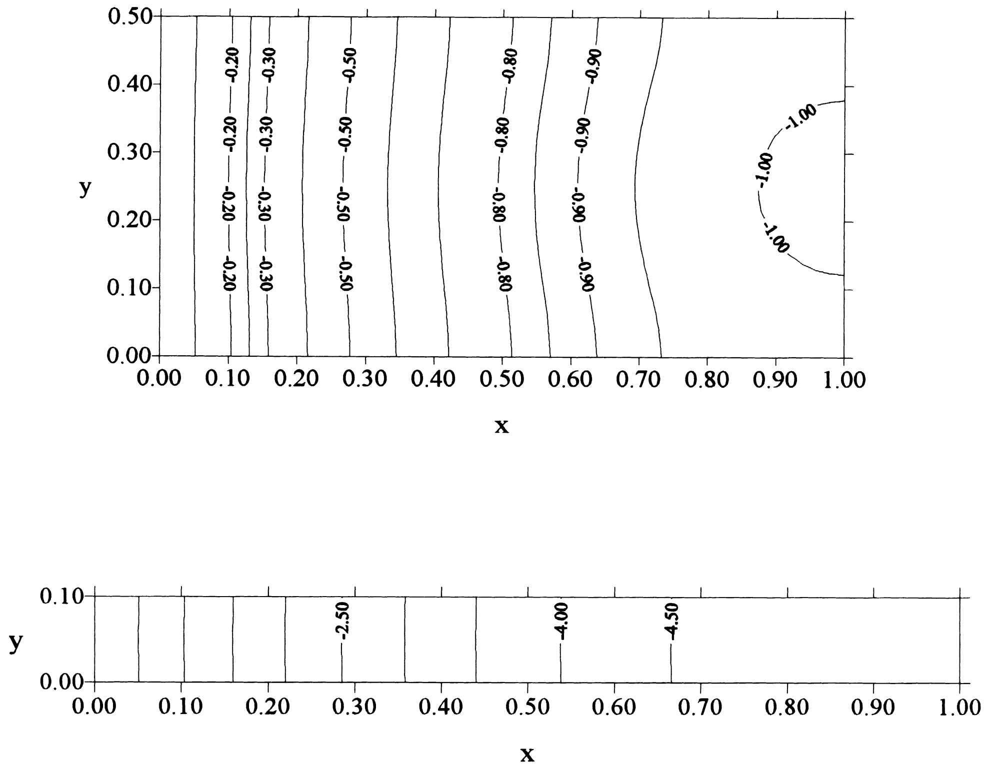

Fig. 8. Dimensionless temperature distributions in a rectangular duct with $\beta = 0.5$ and 0.1, version 1S.

to Schmidt and Newell [9]. If one considers that the Nusselt numbers predicted by Schmidt and Newell [9] are calculated numerically it is possible to appreciate the very high precision of these numerical results. The highest values for Nu are observed for version 2L as for the H2 boundary condition as shown in [5]. For versions 2S and 1S the Nusselt numbers increase with β, while the opposite happens for versions 4, 3L, 2L, 2C and 1L. For version 3S there exists one point of minimum near $\beta = 0.5$: this fact is confirmed by the results quoted in [17] with reference to the uniform velocity profile (slug flow). It is interesting to underline that the well known results of the infinite parallel plates with uniform heat flux on the walls are easily obtained as a limiting situation of rectangular ducts with the aspect ratio approaching zero (for versions 3S, 2C, 1L $Nu = 140/26$ when $\beta \to 0$, while for versions 4, 3L, 2L $Nu = 140/17$ when $\beta \to 0$). The analytical results, in terms of 2D temperature maps and Nusselt numbers, quoted in this paper can be used as a benchmark for commercial and self-produced codes for the investigation of the forced convection of incompress ible laminar fully developed flows inside the ducts of arbitrary cross-section. Finally, a fifth order poly-

Table 9
Polynomial coefficients appearing in Eq. (21)

Version	g_0	g_1	g_2	g_3	g_4	g_5	ε (%)
4	8.235	−16.819	25.327	−20.079	8.3064	−1.3622	0.031
3L	8.235	−13.496	16.839	−10.235	1.6157	0.609	−0.039
3S	5.385	−14.37	35.857	−45.236	30.427	−8.4936	0.289
2L	8.235	−10.073	10.713	−4.8181	−1.2915	1.3301	−0.035
2S	0	11.639	−21.894	30.454	−23.203	7.0994	0.11
2C	5.385	−10.286	18.16	−17.859	9.4777	−2.0673	−0.053
1L	5.385	−6.4026	8.2377	−7.4582	3.5329	−0.609	−0.053
1S	0	5.9624	−7.4114	7.8682	−5.2236	1.4904	−0.097

nomial approximation for calculating the Nusselt numbers for each H1 thermal version is given, with the aim of offering a very simple but accurate tool for technicians and designers involved in heat transfer applications when the aspect ratio of the duct ranges between 0.1 and 1:

$$Nu = \sum_{i=0}^{5} g_i \beta^i \qquad (21)$$

The coefficients g_i are given in Table 9 for all the eight versions of H1 conditions considered; the relative difference ε is positive when Eq. (21) gives the Nusselt numbers greater than the rigorous calculation.

5. Concluding remarks

The paper contains an analytical study of heat transfer for the dynamic and thermal fully developed region of rectangular ducts, for the H1 boundary conditions. The 2D temperature distribution has been analytically determined for all the different combinations of heated and adiabatic walls of practical interest. The Nusselt numbers are accurately predicted and compared with the results obtained numerically by several authors.

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